§14.1: Multivariable Functions

09/27/21

<u>Definition</u>: A multivariable function (of n-variables w/ real values) is a function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$

$n \leftarrow$ # of variables

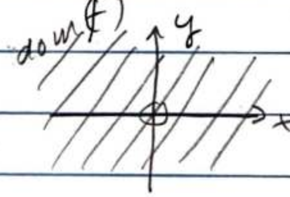
f: function's name; D: function's domain; $\mathbb{R}$: output real #

$\text{dom}(f)$ = domain of f

$\text{ran}(f) = \{ f(\vec{x}) : \vec{x} \in \text{dom}(f) \}$

<u>NB</u>: often, we won't explicitly state the domain of a function given formulaically. We'll use "the natural domain" in that case, i.e. the set of all inputs w/ defined outputs given by the formula

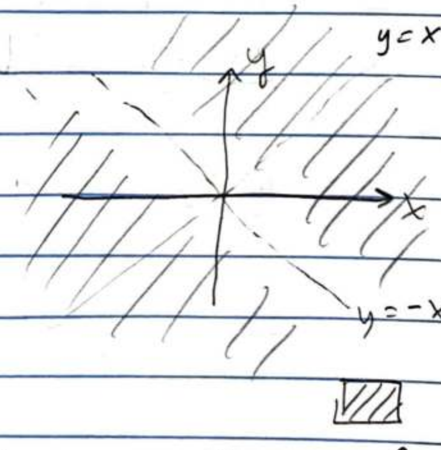
<u>Ex</u>: $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$



$\text{dom}(f) = \{ (x,y) \in \mathbb{R}^2 : \dfrac{x^2-y^2}{x^2+y^2} \text{ is defined} \}$

$= \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0 \} = \{ (x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0) \}$

<u>Ex</u>: $f(x,y) = \dfrac{x^2+y^2}{x^2-y^2}$



$\text{dom}(f) = \{ (x,y) : \dfrac{x^2+y^2}{x^2-y^2} \text{ is defined} \}$

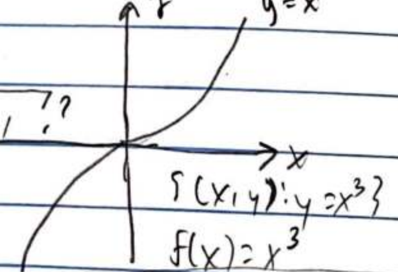
$= \{ (x,y) \in \mathbb{R}^2 : x^2-y^2 \neq 0$

$= \{ (x,y) : x \neq \pm y \}$

$= \{ (x,y) \in \mathbb{R}^2 : |x| \neq |y| \}$

<u>Definition</u>: The <u>graph</u> of a function f is

$\text{graph}(f) = \{ (\vec{x}, f(\vec{x})) : \vec{x} \in \text{dom}(f) \}$



$\boxed{(*)\ \{(x, f(x)) : x \in \text{dom}(f)\}}$

<u>Ex</u>: What is the shape of $f(x,y) = \sqrt{x^2+y^2+1}$??

<u>Sol</u>: Setting $z = f(x,y)$

$z = \sqrt{x^2+y^2+1}$, i.e. $z^2 = x^2+y^2+1$ & $z \geq 0$

i.e. $-x^2-y^2+z^2=1$ & $z \geq 0$

$\uparrow$ two-sheet hyperboloid

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i.e. $-x^2-y^2+z^2=1$ & $z \geq 0$

↖ two-sheet hyperboloid

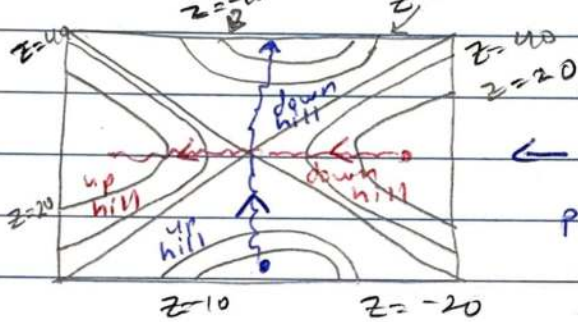
∴ graph(f) is the upper sheet of a two-sheet hyperboloid ▨

<u>Question</u>: How can we represent a two-variable function in 2-space?

<u>Answer</u>: Build a contour map (aka <u>level curves</u>, or <u>elevation map</u>)

<u>Ex</u>:

hyperbolic paraboloid →

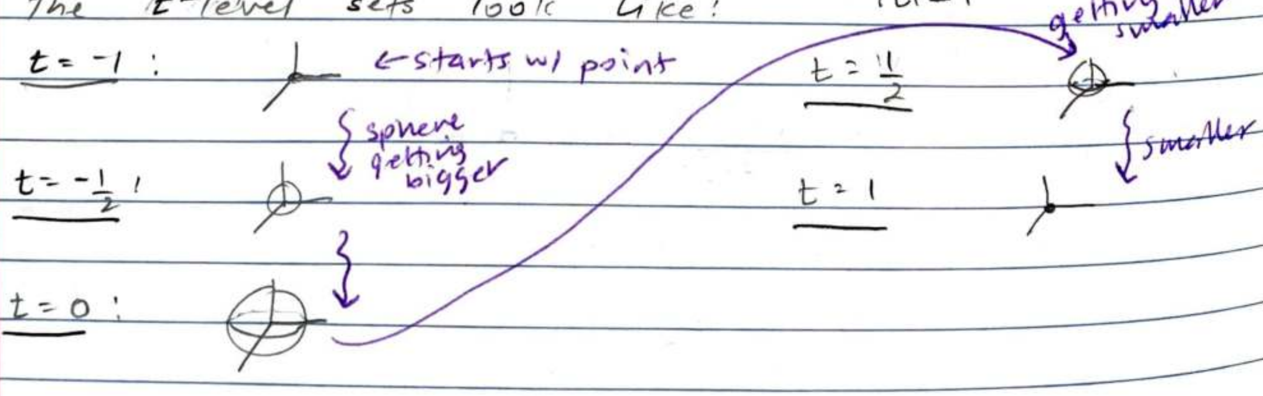


← contour map of hyperbolic paraboloid

<u>Ex</u>: The unit <u>hypersphere</u> is:

$S^3 = \{(x,y,z,t) \in \mathbb{R}^4 : x^2+y^2+z^2+t^2 = 1\}$ ↖ $|t| \leq 1$

The t-level sets look like:



~~No~~ Notation:

$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$

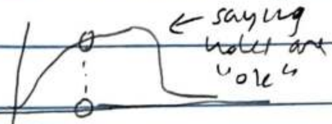
or

$f(\vec{x}) \to L$ as $\vec{x} \to \vec{a}$

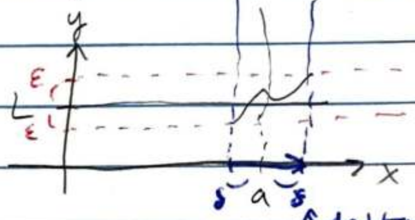
§14.2: Limits & Continuity!

In calculus II, the formal definition of limits is:

~~Definition: Let f be a function & let "a" be a limit point of dom(f)~~

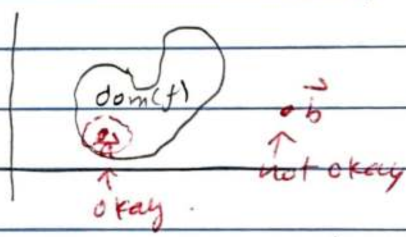


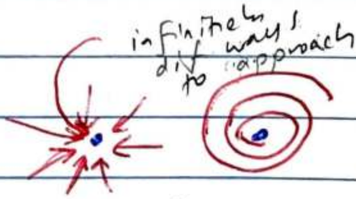
Definition: Let f be a multivariable function & let $\vec{a} \in \mathbb{R}^n$ be a limit point of dom(f). The limit as $\vec{x}$ tends to $\vec{a}$ of f is $L \in \mathbb{R}$ when for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\vec{a} \neq \vec{x} \in$ dom(f) we have $|\vec{x} - \vec{a}| < \delta$ implies $|f(\vec{x}) - L| < \varepsilon$



"calculus III version of "one-sided limits are ~~for~~ equal""

Proposition (Curves Criterion): Let f be a function & $\vec{a}$ a limit point @ its domain. $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ if & only if for all curves $\vec{r}(t)$ in dom(f) such that (s.t.) $\lim_{t \to 0^+} \vec{r}(t) = \vec{a}$ we have $\lim_{t \to 0^+} f(\vec{r}(t)) = L$.





Ex! Show $\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist

Solution! Let $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$ & $l_{ab}(t) = \langle at, bt \rangle$

Note that $\lim_{t \to 0} l_{a,b}(t) = \langle 0, 0 \rangle$

for all $t \neq 0$, we have $f(l_{a,b}(t)) = \frac{(at)^2 - (bt)^2}{(at)^2 + (bt)^2}$

$= \frac{(a^2 - b^2)t^2}{(a^2 + b^2)t^2} = \frac{a^2 - b^2}{a^2 + b^2}$ $\therefore \lim_{t \to 0} f(l_{0,1}(t)) = \lim_{t \to 0} \frac{a^2 - b^2}{a^2 + b^2} \Big|_{a=0, b=1}$

$= \frac{0 - 1}{0 + 1} = -1$, check $\lim_{t \to 0} f(l_{1,1}(t)) = 0 \neq -1$ $\therefore$ by the curves criterion $\lim_{\vec{x} \to 0} f(\vec{x})$ does not exist. $\boxed{/\!/}$